Sasaki-Einstein metrics on spheres

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§ Introduction

(M, g) cpt Riemannian manifold, dim M odd

(M, g) is **Sasakian** if $\left(C(M) = M \times \mathbb{R}_{>0}, \bar{g} = r^2 g + dr^2\right)$
r is the coordinate on $\mathbb{R}_{>0}$ is Kähler

(M, g) is **Sasaki-Einstein** if $\bar{g}$ is Kähler and Ricci flat
$\Updownarrow$
g is Einstein

Today: on spheres

Example: (S^{2n-1}, g_E) is Sasaki-Einstein

$$C(S^{2n-1}) \simeq \mathbb{C}^n - \{0\}$$

Homotopy sphere: Σ^n real cpt n-dimensional manifold (homeo $\Leftrightarrow$) homotopy equiv. to S^n.

Boyer - Galicki - Kollár '05: $\exists$ several SE metrics on any homotopy sphere Σ^{4n+1} that bounds parallelizable manifolds.

Conj: All odd dimensional homotopy spheres that bound parall. manifolds admits SE metic.

Collins - Székelyhidi '17 $\exists$ ∞-many SE metrics on the standard S^5

Conj. $\exists$ ∞-many SE metrics on every standard S^{2n-1}, $n \geq 3$

Thm (–) Any homotopy sphere Σ^{2n-1} that bounds parall. manifolds admits ∞-many SE-metrics, $n \geq 3$.

§ The construction

$n \geq 3$, $a = (a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+1}$, $a_i > 1$

Let $Y(a) = \{ z_0^{a_0} + z_1^{a_1} + \dots + z_n^{a_n} = 0 \} \subseteq \mathbb{C}^{n+1}$

Brieskorn - Pham singularity.

Facts: The link $L(a) = Y(a) \cap S_\varepsilon^{2n+1}$ is a smooth compact simply connected $(2n-1)$-manifold that bounds a parall. manifold.

Milnor '68

Let $d = \mathrm{lcm}(a_i)$, $d_i = \frac{d}{a_i}$

$$\begin{array}{ccc} \mathbb{C}^* \curvearrowright Y(a) & \subseteq & \mathbb{C}^{n+1} \setminus \{0\} \\ \downarrow & & \downarrow \\ X^{orb} & \subseteq & \mathbb{P}(d_0, \dots, d_n) \end{array} \qquad \lambda \cdot (z_0, \dots, z_n) = (\lambda^{d_0} z_0, \dots, \lambda^{d_n} z_n)$$

Fact (BG) $L(a)$ admits a SE metric iff X^{orb} admits a Fano KE-metric.

Thm (–) Assume $a_0 \leq a_1 \leq \dots \leq a_n$. Then X^{orb} admits a KE-metric iff $1 < \sum_i \frac{1}{a_i} < 1 + \frac{n}{a_n}$.

Rmk: The condition (Lichnerowicz obstruction) well known to be necessary.

Proof. X^{orb} Fano iff $d - \sum d_i < 0 \iff 1 < \sum \frac{1}{a_i}$

Let (X, Δ) be a pair associated to X^{orb}:

$g_j = \gcd(d_0, \dots, \hat{d}_j, \dots, d_n)$, $a_i' = a_i / g_i$, $g = g_0 \cdots g_n$

$X \simeq \{ z_0^{e_0'} + \dots + z_n^{e_n'} \} \subseteq \mathbb{P}\left(\frac{d_0 g_0}{g}, \dots, \frac{d_n g_n}{g} \right) =: \mathbb{P}'$

well formed

$\Delta = \sum_i \left(1 - \frac{1}{g_j}\right) H_j$

$H_j = \{ z_j = 0 \} \cap X$

Thm (Collins–Székelyhidi) X^{orb} admits a KE-metric iff (X, Δ) is K-polystable.

Take $\pi : \mathbb{P}^1 \to \mathbb{P}^n$ $\quad [z_0 : \ldots z_n] \mapsto [z_0^{e_0} : \ldots z_n^{e_n}]$

$$K_X + \Delta_X = \pi^*\left(K_2 + \sum\left(1 - \frac{1}{e_i}\right)L_i\right)$$ where

$L = \{w_0 + \ldots + w_n = 0\}$, $\quad L_i = L \cap \{w_i = 0\}$

Zhuang : (X, Δ) is K-polystable iff

$\left(L, \sum\left(1 - \frac{1}{e_i}\right)L_i\right)$ is K-poly.

Fujita : tells you when hyperplane arrang. are K-poly.

$\square$

§ Homotopy spheres $(n \geq 3)$

Kervaire - Milnor :

$$\Theta_{2n-1} = \left\{ \begin{array}{l} \text{homotopy spheres} \\ \text{of dim } 2n-1 \end{array} \right\} \Big/ \begin{array}{l} \text{oriented} \\ \text{diff.} \end{array}$$

finite ab group

$$\vee$$

bP_{2n} : classes of spheres that bound parall. manifold

$n = 2m+1$ odd $\qquad bP_{4m+2}$ is either 0 or $\mathbb{Z}_2$

$n = 2m$ even $\qquad bP_{4m+2}$ is cyclic of order

$$|bP_{4m+2}| \sim 2^{4m}$$

Rmk: $\Theta_7 = bP_8$

Recall: $Y(a) = \{ z_0^{a_0} + \dots + z_n^{a_n} = 0 \}$, $L(a)$ link

$G(a)$ graph: $n+1$ vertices a_i

a_i and a_j are connected $(i \neq j)$ iff

$\gcd(a_i, a_j) \neq 1$.

Thm (Brieskorn) If $G(a)$ contains at least two isolated vertices, then $L(a)$ is a homotopy sphere.

$n = 2m$ even Assume that $L(a) \in bP_{4m}$. The diffeomorphism type of $L(a)$ is determined by

$\frac{1}{8}\tau(e) \mod |bP_{4m}|$ where

$\tau(e)$ has a combinatorial expression depending on e_i.

Brieskorn spheres: $e = (2, 2, \ldots, 2, 3, 6k \pm 1)$, $n = 2m$

$L(e) \in bP_{4m}$, $\frac{\tau(e)}{8} = (-1)^m k$, so

all elements in bP_{4m} are taken.

Our condition: $1 < \sum \frac{1}{e_i} < 1 + \frac{n}{e_n}$

Our examples: Take $k \in \{1, \dots, |bP_{4m}|\}$ and

$$e_0 = e_1 = 2, \quad e_2 = \dots e_{n-2} = p, \quad e_{n-1} = p+1$$
$$e_n = p + \ell$$

where $\ell = 6k-3$, $p \equiv 2 \mod m\ell(\ell-1)|bP_{4m}|$

For $p \gg 0$, $L(e_p)$ admits SE-metrics

Proposition: $\tau(e_p)$ is a polynomial in p

$$\Rightarrow \tau(e_p) \equiv (-1)^m k \mod |bP_{4m}| .$$